

# Four-loop anomalous dimensions in Leigh-Strassler deformations

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**Abstract.** We determine the scalar part of the four-loop chiral dilatation operator for Leigh-Strassler deformations of  $\mathcal{N} = 4$  super Yang-Mills. This is sufficient to find the four-loop anomalous dimensions for operators in closed scalar subsectors. This includes the  $SU(2)$  subsector of the (complex)  $\beta$ -deformation, where we explicitly compute the anomalous dimension for operators with a single impurity. It also includes the “3-string null” operators of the cubic Leigh-Strassler deformation. Our four-loop results show that the rational part of the anomalous dimension is consistent with a conjecture made in arXiv:1108.1583 based on the three-loop result of arXiv:1008.3351 and the  $\mathcal{N} = 4$  magnon dispersion relation. Here we find additional  $\zeta(3)$  terms.

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# 1 Introduction

The spectrum of single trace operators in planar  $\mathcal{N} = 4$  supersymmetric Yang-Mills (SYM) is solvable, at least in principle if not always in practice, because of an underlying integrability (see [1] for a comprehensive review). Starting with  $\mathcal{N} = 4$  SYM, there exists a class of deformations that break the superconformal symmetry down to  $\mathcal{N} = 1$  but preserve the integrability. However, there exist other deformations that destroy the integrability and we can ask ourselves to what extent one can calculate the spectra of these theories. A possible way forward is to compute the dilatation operator to high enough loop order such that a pattern emerges.

The general class of deformations of  $\mathcal{N} = 4$  SYM that preserve an  $\mathcal{N} = 1$  superconformal symmetry were first catalogued by Leigh and Strassler [2]. The Leigh-Strassler superpotential is given by

$$W = i\kappa \left[ \text{tr}(XYZ - qXZY) + \frac{h}{3} \text{tr}(X^3 + Y^3 + Z^3) \right], \quad (1.1)$$

which depends on three complex parameters  $\kappa$ ,  $q$  and  $h$ , although the imaginary part of  $\kappa$  can be eliminated by a chiral field phase rotation. The deformation is exactly marginal if the Yang-Mills coupling  $g_{\text{YM}}$  and the deformation parameters satisfy the relation

$$2g_{\text{YM}}^2 = \kappa\bar{\kappa}(1 + q\bar{q} + h\bar{h}) + \mathcal{O}((\kappa\bar{\kappa})^4), \quad (1.2)$$

where we have allowed for a fourth-order correction, to be discussed in more detail below. This leaves a three complex dimensional space of  $\mathcal{N} = 1$  superconformal theories. Setting the coefficients to  $\kappa = g_{\text{YM}}$ ,  $q = 1$ ,  $h = 0$ , we recover the  $\mathcal{N} = 4$  superpotential

$$W = ig_{\text{YM}} \text{tr}[X, Y]Z. \quad (1.3)$$

The deformation (1.1) includes some interesting special cases. The most well known deformation is the so-called  $\beta$ -deformation (see [3] for a review) where  $q = e^{-2i\pi\beta}$ ,  $h = 0$  with  $\beta$  real. Inserting this into (1.2) then yields  $\kappa = g_{\text{YM}}$  which is exact in the planar limit [4, 5]. With a field redefinition, the  $\beta$ -deformed superpotential can be recast into the form

$$W = ig_{\text{YM}} \text{tr}(e^{i\pi\beta}XYZ - e^{-i\pi\beta}XZY). \quad (1.4)$$

Furthermore, the computation of the spectrum for local operators is an integrable problem [6–9]. In [7] it was shown that at the one-loop level the corresponding Bethe equations are the same as in  $\mathcal{N} = 4$  SYM, except for a  $\beta$ -dependent shift. This was extended to all loops in [8]. The supergravity dual for the  $\beta$ -deformed theory is known [10] and its world-sheet theory has been shown to be classically integrable, even though it is not a coset [9, 11]. Other integrable deformations with nonzero values of  $h$  can be obtained by acting on the  $\beta$ -deformed theories with similarity transformations [12–14].

If the deformation is generalized to complex  $\beta$ , the relation of the couplings has to be altered at four-loop order [15, 16], as indicated in (1.2). The resulting spin-chain is a known integrable model only in a certain subsector [7]. This subsector plays an important role in  $\mathcal{N} = 4$  SYM and in its (complex)  $\beta$ -deformation, where it is closed, at least perturbatively [17]. It consists of operators composed of two flavors of complex

scalar fields. The subsector is called the  $SU(2)$  subsector, since an  $SU(2)$  subgroup of the  $SU(4)$   $R$ -symmetry of  $\mathcal{N} = 4$  transforms the two flavors into each other. We will stick with this name, even if the  $SU(2)$   $R$ -symmetry is broken in the presence of the (complex)  $\beta$ -deformation. In a formulation with manifest  $\mathcal{N} = 1$  supersymmetry, these complex scalars are the lowest components of chiral superfields. Hence, being composites of two of these three superfield flavors, the operators of the  $SU(2)$  subsector are themselves chiral superfields.

There exist a bigger subsector that consists of all possible chiral composite operators. This means, the latter can contain all three types of chiral superfield flavors  $X$ ,  $Y$ ,  $Z$ , and also the chiral superfield strength  $W_\alpha$ . Starting at two loops, three chiral field flavors can transform into two  $W_\alpha$  and *vice versa*. If  $\kappa \neq 0$ ,  $h = 0$  in (1.1), this mixing can only occur if all three interacting chiral field flavors are different, while at  $h \neq 0$  a mixing occurs even if all these flavors are identical. This guarantees that the  $SU(2)$  subsector is closed whenever  $h = 0$ , but it also means that in the case  $h \neq 0$  the mixing extends to the bigger subsector. When restricting to the lowest lying components of the superfields, the field content of this bigger subsector is reduced to the three complexified scalars and the two-component gaugino  $\psi_\alpha$ . In  $\mathcal{N} = 4$  SYM this is the  $SU(2|3)$  subsector [18].

Let us now consider those deformations that have closed scalar subsectors and restrict ourselves to the planar limit. For the (complex)  $\beta$ -deformations, the simplest single-trace operators are  $L$  copies of a single superfield flavor  $X$ ,

$$\text{tr } X \dots X , \quad (1.5)$$

which are protected from quantum corrections. In the spin chain picture each of these operators is the ferromagnetic ground state of a closed spin chain with fixed length  $L$ . One constructs excited states by changing the flavor of one or more of the fields. The composite operator that contains a single changed flavor

$$\text{tr } Y X \dots X \quad (1.6)$$

corresponds to a spin chain with one excitation (magnon). While in the  $\mathcal{N} = 4$  case this operator is still protected, in the (complex)  $\beta$ -deformed theory it acquires an anomalous dimension. We can add more excitations by adding more  $Y$  fields. Adding  $Z$  fields takes us out of the  $SU(2)$  subsector.

The only other way to have a closed scalar subsector is to take the limit  $\kappa \rightarrow 0$  with fixed  $\kappa|h| = \sqrt{2}g_{\text{YM}}$ , such that the superpotential becomes

$$W = i\sqrt{2}g_{\text{YM}} \text{tr}(X^3 + Y^3 + Z^3) . \quad (1.7)$$

This special case, called the (Fermat) cubic Leigh-Strassler deformation, was recently analyzed in [19]. This deformation has no closed subsector resembling the  $SU(2)$  subsector of the undeformed theory. However, because the superpotential does not have the “hopping” term there is no direct mixing of scalar operators in the planar limit. Scalar mixing can still occur indirectly through intermediate scalar-fermion mixing, but if the operator does not have a sequence of three or more identical scalar fields, then this mixing will not occur either. We will call such operators “3-string null”. In the planar limit these operators do not mix with other operators.

In the cubic Leigh-Strassler theory, the ground states are anti-ferromagnetic in nature, having the general form [19]

$$\text{tr } XYXZYZ \dots XY , \quad (1.8)$$

where no two neighboring fields have the same flavor. These operators are protected in the planar limit to all orders in perturbation theory. Operators of the type

$$\text{tr } XYXYYZ \dots XY , \quad (1.9)$$

where there are two neighboring fields with the same flavor correspond to the first excited states. They are not protected, but they are 3-string null and hence do not undergo operator mixing either. The lack of mixing makes it possible to formulate an all-loop conjecture about the rational part of their anomalous dimensions [19]. The conjecture is applicable to any 3-string null operator [19].

The upshot is that we have two classes of Leigh-Strassler theories where the mixing is closed to scalar operators; the  $SU(2)$  subsector of the complex  $\beta$ -deformed theories and the 3-string null operators in the cubic Leigh-Strassler theory. For these two cases we only need the scalar part of the “chiral” dilatation operator, where “chiral” indicates the restriction to the field content of the  $SU(2|3)$  subsector.

In this paper we determine the scalar part of the chiral dilatation operator to four-loop order. The evaluation of the relevant Feynman diagrams becomes manageable by determining the deviations from the four-loop dilatation operator in the  $SU(2)$  subsector of the  $\mathcal{N} = 4$  SYM theory. The latter has been determined [20] as one of the conserved local charges [21,22] using integrability and the postulated form of the magnon dispersion relation. Hence, by doing perturbation theory in the general Leigh-Strassler case, we determine the scalar part of the chiral dilatation operator. These more general deformations are not believed to be integrable. But our result is also valid for the mixing between all three scalar chiral fields of  $\mathcal{N} = 4$  SYM.

The scalar part we find for the chiral dilatation operator is valid for *any* Leigh-Strassler deformation. It is complete when acting on a closed scalar subsector, hence we can apply it to the above cases to find the anomalous dimensions to four-loop order. When acting on the 3-string null operators in the cubic Leigh-Strassler theory we find that the rational contributions to the anomalous dimensions are consistent with an all-loop conjecture made in [19]. Furthermore, we see that the four-loop anomalous dimensions contain additional transcendental terms.

The paper is organized as follows. In section 2 we analyze the Feynman diagrams that modify the scalar part of the chiral four-loop dilatation operator of  $\mathcal{N} = 4$  SYM theory for Leigh-Strassler deformations. In section 3 we simplify the results from the previous section and determine the anomalous dimensions of the one impurity operators in the (complex)  $\beta$ -deformation and the 3-string null operators in the cubic Leigh-Strassler deformation. In section 4 we draw our conclusions. Several technical details concerning the so-called chiral functions, similarity transformations and relevant loop integrals are included in various appendices.

## 2 Feynman diagram analysis

The perturbative spectrum of local composite operators in a conformal field theory is given by the bare dimensions of the operators plus their anomalous dimensions. The latter are generated by the renormalization of the composite operators

$$\mathcal{O}_{a,\text{ren}} = \mathcal{Z}_a{}^b(\lambda, \varepsilon) \mathcal{O}_{b,\text{bare}} , \quad (2.1)$$

which in general imply a mixing among the operators.  $\mathcal{Z}$  is the matrix-valued renormalization constant that is given as a power series in the 't Hooft coupling constant  $\lambda = g_{\text{YM}}^2 N$  and absorbs the overall UV divergences which are generated at each loop order. In  $D = 4 - 2\varepsilon$  dimensions the divergences appear as poles in  $\varepsilon$ .

We determine the anomalous dimensions as eigenvalues of the dilatation operator, which in terms of  $\mathcal{Z}$  is defined as

$$\mathcal{D} = \mu \frac{d}{d\mu} \ln \mathcal{Z}(\lambda \mu^{2\varepsilon}, \varepsilon) = \lim_{\varepsilon \rightarrow 0} \left[ 2\varepsilon \lambda \frac{d}{d\lambda} \ln \mathcal{Z}(\lambda, \varepsilon) \right] . \quad (2.2)$$

The logarithm in the above description is understood as a series expansion in the orthonormalized basis of composite operators for the unrenormalized theory, such that the leading contribution to  $\mathcal{Z}$  is the identity. All higher order poles in  $\ln \mathcal{Z}$  must cancel, leading to the second relation in (2.2). The anomalous dimensions are then the eigenvalues of  $\mathcal{D}$ .

We first calculate the renormalization constant  $\mathcal{Z}$  for operators composed of scalar chiral superfields by computing the relevant Feynman diagrams using an  $\mathcal{N} = 1$  superfield formulation. We then use (2.2) to derive the dilatation operator. For a more detailed description we refer the reader to [23, 24]. It should be understood that  $\mathcal{D}$  is the scalar part of the chiral dilatation operator, since we have restricted ourselves to scalar chiral operators. The dilatation operator can be expressed as the series expansion

$$\mathcal{D} = \sum_{n=1}^{\infty} g^{2n} \mathcal{D}_n , \quad g \equiv \frac{\sqrt{\lambda}}{4\pi} . \quad (2.3)$$

For  $\mathcal{N} = 4$  SYM, the first three terms in the expansion were found to be

$$\begin{aligned} \mathcal{D}_1 &= -2\chi(1) , \\ \mathcal{D}_2 &= -2[\chi(1, 2) + \chi(2, 1)] + 4\chi(1) , \\ \mathcal{D}_3 &= -4(\chi(1, 2, 3) + \chi(3, 2, 1)) + 4i\epsilon_2[\chi(2, 1, 3) - \chi(1, 3, 2)] - 4\chi(1, 3) \\ &\quad + 16(\chi(1, 2) + \chi(2, 1)) - 16\chi(1) - 4(\chi(1, 2, 1) + \chi(2, 1, 2)) , \end{aligned} \quad (2.4)$$

where  $\epsilon_2 = -\frac{i}{2}$  is a parameter that can be changed by similarity transformations of the basis of operators.

The terms in (2.4) are expressed using the very convenient basis of so-called chiral functions. They capture the structure of the chiral and anti-chiral superfields within the Feynman diagrams [25, 26] (see [23] for a review). Here they are defined as

$$\chi(a_1, \dots, a_n) = \sum_{r=0}^{L-1} \prod_{i=1}^n F_{a_i+r \ a_i+r+1} . \quad (2.5)$$



$$\begin{aligned}
(\mathbf{P}-\mathbf{1})_{ij} &= \frac{1}{4}(1+q\bar{q})(\lambda_i^3\lambda_j^3+\lambda_i^8\lambda_j^8-2\lambda_i^0\lambda_j^0)+\frac{\sqrt{3}}{4}(1-q\bar{q})(\lambda_i^8\lambda_j^3-\lambda_i^3\lambda_j^8) \\
&\quad +q(\lambda_i^{-1}\lambda_j^{+1}+\lambda_i^{+2}\lambda_j^{-2}+\lambda_i^{-3}\lambda_j^{+3})+\bar{q}(\lambda_i^{+1}\lambda_j^{-1}+\lambda_i^{-2}\lambda_j^{+2}+\lambda_i^{+3}\lambda_j^{-3})\;, \\
(\epsilon\mathbf{d})_{ij} &= \lambda_i^{+1}\lambda_j^{-3}+\lambda_i^{-2}\lambda_j^{-1}+\lambda_i^{+3}\lambda_j^{+2}-\bar{q}(\lambda_i^{-1}\lambda_j^{-2}+\lambda_i^{+2}\lambda_j^{+3}+\lambda_i^{-3}\lambda_j^{+1})\;, \\
(\mathbf{d}\epsilon)_{ij} &= \lambda_i^{-1}\lambda_j^{+3}+\lambda_i^{+2}\lambda_j^{+1}+\lambda_i^{-3}\lambda_j^{-2}-q(\lambda_i^{+1}\lambda_j^{+2}+\lambda_i^{-2}\lambda_j^{-3}+\lambda_i^{+3}\lambda_j^{-1})\;, \\
\mathbf{d}\mathbf{d}_{ij} &= \frac{1}{2}(\lambda_i^3\lambda_j^3+\lambda_i^8\lambda_j^8+\lambda_i^0\lambda_j^0)\;.
\end{aligned} \tag{2.9}$$
$$\lambda^{\pm 3} \rightarrow \sigma^{\pm}, \quad \lambda^3 \rightarrow \sigma^3, \quad \lambda^8 \rightarrow \frac{1}{\sqrt{3}} \mathbb{1}, \quad \lambda^0 \rightarrow \sqrt{\frac{2}{3}} \mathbb{1}, \quad (2.10)$$

## 2.1 Self energy diagrams

$$\text{---}\bigcirc\text{---} = N\kappa\bar{\kappa}(1 + q\bar{q} + h\bar{h})I_1 \ , \quad \text{---}\overbrace{\hspace{0.8cm}}^{\text{wavy}}\text{---} = -2Ng_{\text{YM}}^2 I_1 \ , \quad (2.11)$$

At four loops, this is no longer the case and leads to a modification of the relation in (1.2). Divergences no longer cancel at the same loop-level, but the four-loop contribution to the anomalous dimension can be cancelled by a one-loop contribution if (1.2) is modified by a fourth-order term. This was previously worked out for the (complex)  $\beta$ -deformation in [15, 16] and for the entire Leigh-Strassler deformation (1.1) in [27]. Here, we recall these results.

$$\text{---} \circlearrowleft \text{---} \rightarrow -N^4 \frac{(\kappa\bar{\kappa})^4}{2} ((1 + q\bar{q} + h\bar{h})^4 + \delta H(q, \bar{q}, h, \bar{h})) I_{4t} , \quad (2.12)$$

where the first term is cancelled by the remaining four-loop diagrams. The additional contribution is a straightforward generalization of the result at  $h = 0$  in [15, 16] and in accord with [27] we find

$$\begin{aligned} \delta H(q, \bar{q}, h, \bar{h}) = & (1 - q\bar{q} - h\bar{h})^4 + 8q\bar{q}h\bar{h}(q\bar{q}(3 - q\bar{q}) + h\bar{h}(3 - h\bar{h})) \\ & + 8q\bar{q}(h^3 + \bar{h}^3) - 8h\bar{h}(q^3 + \bar{q}^3) - 8(q^3\bar{h}^3 + \bar{q}^3h^3) . \end{aligned} \quad (2.13)$$

Since this term is not cancelled by the other four-loop contributions it will generate an anomalous dimension for the chiral field, unless one allows for fourth order corrections to the marginal couplings. Unlike in the complex  $\beta$ -deformed case, it is not sufficient to have a series expansion with constant coefficients for the correction in (1.2). Instead, we should make the ansatz

$$\kappa \rightarrow \kappa(1 + \lambda^3 \Delta\kappa(q, \bar{q}, h, \bar{h})) , \quad \bar{\kappa} \rightarrow \bar{\kappa}(1 + \lambda^3 \Delta\bar{\kappa}(q, \bar{q}, h, \bar{h})) . \quad (2.14)$$

If we absorb these corrections into the original  $\kappa, \bar{\kappa}$  the relation (1.2) is modified to

$$2g_{\text{YM}}^2 = \kappa\bar{\kappa}(1 + q\bar{q} + h\bar{h})(1 - \lambda^3(\Delta\kappa + \Delta\bar{\kappa})) , \quad (2.15)$$

where further modifications will occur at higher orders. With these corrections, the leftover piece (2.13) from the four-loop diagrams can be cancelled by the chiral one-loop diagram (2.11) with modified couplings (2.14). For conformal invariance we have to cancel the anomalous dimensions which is achieved by setting

$$2 \left[ \text{blob} \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \text{blob} \right] + 8 \left[ \text{---} \bigcirc \text{---} + \dots \right] = \text{finite} , \quad (2.16)$$

where the blobs are the vertex corrections which come with factors of  $\lambda^3 \Delta\kappa$  or  $\lambda^3 \Delta\bar{\kappa}$ . The ellipsis denotes the remaining four-loop diagrams in which the chiral field lines form (nested) bubbles. The integer prefactors come from the definition of the dilatation operator (the anomalous dimension) in (2.2). This leads to a multiplication of the  $K$ -loop contribution by a factor  $2K$ . From the condition (2.16) we find

$$\Delta\kappa + \Delta\bar{\kappa} = 16 \frac{\delta H(q, \bar{q}, h, \bar{h})}{(1 + q\bar{q} + h\bar{h})^4} \frac{\mathcal{I}_{4t}}{\mathcal{I}_1} . \quad (2.17)$$

With this adjustment the theory is conformal but not finite, since the diagrams that contribute to the chiral field renormalization do not have the prefactors in (2.16) and hence do not cancel.

The alterations induced by (2.12) also affect the operator renormalization. At four-loop order this only concerns the diagrams associated with the simplest chiral function  $\chi(1)$ . But the relation in (2.16) guarantees that these diagrams obey

$$2 \left[ \text{blob} \text{---} \text{---} \text{---} + \text{---} \text{---} \text{blob} \right] + 8 \left[ \text{---} \text{---} \text{---} + \dots \right] = \text{finite} , \quad (2.18)$$



whose validity is obvious if one realizes that the chiral operator (represented by the bold line) can be treated like a chiral vertex. Hence, the diagrams in (2.18) have the same topology as those in (2.16) and so have the same cancellation of UV divergences. These diagrams can then be omitted from the very beginning since they will not contribute to the dilatation operator.

## 2.2 Diagrams with reducible chiral functions

In the  $SU(2)$  subsector of the (complex)  $\beta$ -deformed theories, not all of the chiral functions in (2.5) are independent. Many chiral functions contain terms that antisymmetrize three neighboring sites. Such terms are zero when there are only two flavors, leading to relations between different chiral functions. The simplest example is

$$\chi(1, 2, 1)|_{SU(2)} = \chi(2, 1, 2)|_{SU(2)} = \rho^4 q \bar{q} \chi(1)|_{SU(2)} , \quad (2.19)$$

where  $|_{SU(2)}$  indicates the projection onto the subsector. The general set of relations are worked out in appendix A. The chiral functions  $\chi(1, 2, 1)$  and  $\chi(2, 1, 2)$  first appear in the three-loop dilatation operator  $\mathcal{D}_3$ . The respective terms in (2.4) arise from the chiral Feynman diagram

$$\begin{array}{c} \text{Diagram: A vertical line with two loops on the left and right, and a horizontal line connecting them in the middle. The bottom line is bolded.} \end{array} = \lambda^3 I_3 \chi(1, 2, 1) , \quad (2.20)$$

and its reflection. At four loops, the relation (2.19) generalizes to chiral functions with four arguments. Here there are two types of relations, given by

$$\begin{aligned} \chi(1, 2, 1, 2)|_{SU(2)} &= \chi(1, 2, 3, 2)|_{SU(2)} = \chi(2, 1, 2, 3)|_{SU(2)} = \rho^4 q \bar{q} \chi(1, 2) , \\ \chi(2, 1, 2, 1)|_{SU(2)} &= \chi(3, 2, 1, 2)|_{SU(2)} = \chi(2, 3, 2, 1)|_{SU(2)} = \rho^4 q \bar{q} \chi(2, 1) , \\ \left. \begin{aligned} \chi(1, 2, 1, 3)|_{SU(2)} &= \chi(1, 3, 2, 3)|_{SU(2)} \\ \chi(3, 2, 3, 1)|_{SU(2)} &= \chi(3, 1, 2, 1)|_{SU(2)} \end{aligned} \right\} &= \rho^4 q \bar{q} \chi(1, 3) , \end{aligned} \quad (2.21)$$

where a relation and its reflection are displayed in tandem. We call chiral functions “reducible” if they simplify as in (2.19) and (2.21) when projected onto the  $SU(2)$  subsector.

In analogy to the three-loop diagram (2.20), four-loop diagrams that come with chiral functions having four arguments are chiral. Hence, each one is generated by one diagram. After D-algebra, we find

$$\begin{aligned} \begin{array}{c} \text{Diagram: A vertical line with two loops on the left and right, and a horizontal line connecting them in the middle. The bottom line is bolded.} \end{array} &= \lambda^4 I_4 \chi(1, 2, 1, 2) , & \begin{array}{c} \text{Diagram: A vertical line with two loops on the left and right, and a horizontal line connecting them in the middle. The bottom line is bolded.} \end{array} &= \lambda^4 I_{4w} \chi(1, 2, 3, 2) , & \begin{array}{c} \text{Diagram: A vertical line with two loops on the left and right, and a horizontal line connecting them in the middle. The bottom line is bolded.} \end{array} &= \lambda^4 I_4 \chi(2, 1, 2, 3) , \\ \begin{array}{c} \text{Diagram: A vertical line with two loops on the left and right, and a horizontal line connecting them in the middle. The bottom line is bolded.} \end{array} &= \lambda^4 I_{4bb} \chi(1, 2, 1, 3) , & \begin{array}{c} \text{Diagram: A vertical line with two loops on the left and right, and a horizontal line connecting them in the middle. The bottom line is bolded.} \end{array} &= \lambda^4 I_{4w} \chi(1, 3, 2, 3) , \end{aligned} \quad (2.22)$$

as well as analogous results for their reflections.

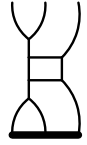
There are also four-loop diagrams having chiral function  $\chi(1, 2, 1)$  or  $\chi(2, 1, 2)$ . These diagrams are constructed by attaching a vector propagator to (2.20) or its reflection. Since the one-loop chiral self energy is identically zero, the ends of the vector propagator must attach to different chiral propagators. Moreover, there are further restrictions coming from the finiteness conditions of [24]. First, the one chiral vertex in (2.20) which is not part of a loop must remain out of any loop after adding the vector field interaction. Second, the vector propagator cannot attach to the neighboring field lines of (2.20), keeping the range of the diagram to three sites. The diagrams that fulfill these constraints, along with their values, are

$$\begin{aligned}
& \text{Diagram 1} , \text{Diagram 2} , \text{Diagram 3} , \text{Diagram 4} , \text{Diagram 5} , \text{Diagram 6} , \text{Diagram 7} , \text{Diagram 8} \rightarrow -I_4 , \\
& \text{Diagram 9} , \text{Diagram 10} \rightarrow -I_{4w} , \quad \text{Diagram 11} , \text{Diagram 12} \rightarrow -I_{4bt} , \quad \text{Diagram 13} \rightarrow -I_{43t} , \\
& \text{Diagram 14} , \text{Diagram 15} , \text{Diagram 16} , \text{Diagram 17} , \text{Diagram 18} \rightarrow I_4 , \quad \text{Diagram 19} \rightarrow I_{4bt} , \quad (2.23) \\
& \text{Diagram 20} \rightarrow I_4 + I_{4\beta} - I_{4tr1} , \quad \text{Diagram 21} \rightarrow I_4 + I_{4w} - I_{4tr2} , \\
& \text{Diagram 22} \rightarrow -I_{4w} , \quad \text{Diagram 23} \rightarrow -I_4 ,
\end{aligned}$$

where we have grouped together the diagrams that lead to the same integrals after D-algebra. The integrals are listed in (C.1). The above results contain all signs from color and flavor factors, and from the D-algebra manipulations. Finite contributions and a common factor  $\lambda^4 \chi(1, 2, 1)$  have been omitted. Note that most of the diagrams are easily evaluated using the arguments in [24]. The diagrams in the fourth line require a little more work and lead to integrals with momenta in the numerator of their integrands that are contracted as prescribed by a trace over products of  $\gamma$ -matrices. Using the results of appendix C, we find for the sum of the most complicated diagrams

$$\text{Diagram 20} + \text{Diagram 21} \rightarrow -2I''_{4t1} + I_{4w} + I_{4bt} + I_{43t} - I_{4t} . \quad (2.24)$$

Summing up all diagrams of (2.23), and eliminating  $I_{4w}$  by making use of the relation (C.4) results in

$$
+ \text{wavy line} = \lambda^4 (-6I_4 + 2I''_{4t1} + I_{4t}) \chi(1, 2, 1) . \quad (2.25)$$

The result for the reflected diagrams is obtained by replacing  $\chi(1, 2, 1)$  with  $\chi(2, 1, 2)$ .

### 2.3 Four-loop result

The reducible contribution to the renormalization constant is the negative sum of (2.25), (2.22) and their reflections. Using (2.2), one finds that the reducible part contributes to the dilatation operator with the coefficient of the  $\frac{1}{\epsilon}$  pole multiplied by 8. Inserting the explicit expressions in (C.1) then gives

$$\begin{aligned}
\delta\mathcal{D}_{4\text{red}} = & (60 - 8\zeta(3))[\chi(1, 2, 1) + \chi(2, 1, 2)] \\
& - 10[\chi(1, 2, 1, 2) + \chi(2, 1, 2, 1)] \\
& - (10 - 8\zeta(3))[\chi(1, 2, 3, 2) + \chi(3, 2, 1, 2)] \\
& - 10[\chi(2, 1, 2, 3) + \chi(2, 3, 2, 1)] \\
& + \left(\frac{10}{3} - 4\zeta(3)\right)[\chi(1, 2, 1, 3) + \chi(3, 2, 3, 1)] \\
& - (10 - 8\zeta(3))[\chi(1, 3, 2, 3) + \chi(3, 1, 2, 1)] . \quad (2.26)
\end{aligned}$$

Restricting to the  $SU(2)$  subsector of  $\mathcal{N} = 4$  SYM and using the identities in (2.19) and (2.21), reduces the above term to

$$\begin{aligned}
\delta\mathcal{D}_{4\text{red}}|_{SU(2)} = & 8(15 - 2\zeta(3)) \chi(1) - 2(15 - 4\zeta(3))[\chi(1, 2) + \chi(2, 1)] \\
& - 8\left(\frac{5}{3} - \zeta(3)\right) \chi(1, 3) . \quad (2.27)
\end{aligned}$$

The four-loop dilatation operator  $\mathcal{D}_{4, \mathcal{N}=4}$  for the  $SU(2)$  subsector of  $\mathcal{N} = 4$  SYM was first presented in [20]. It is determined as one of the commuting charges of the integrable system, using information from the all loop Bethe equations [28] and the magnon dispersion relation [29–31]

$$E(p) = \sqrt{1 + 4h^2(g) \sin^2 \frac{p}{2}} - 1 , \quad (2.28)$$

where we assume that  $h^2(g) = 4g^2$ . To find  $\mathcal{D}_4$  for the Leigh-Strassler theories we add (2.26) to  $\mathcal{D}_{4, \mathcal{N}=4}$  and then subtract (2.27). Using the convention in [26] for  $\mathcal{D}_{4, \mathcal{N}=4}$ , we

find

$$\begin{aligned}
\mathcal{D}_4 = & + 16(5 + \zeta(3)) \chi(1) \\
& - 8(15 + \zeta(3))[\chi(1, 2) + \chi(2, 1)] + 8\left(\frac{23}{3} - \zeta(3)\right) \chi(1, 3) - 4\chi(1, 4) \\
& + 4(15 - 2\zeta(3))[\chi(1, 2, 1) + \chi(2, 1, 2)] + 60[\chi(1, 2, 3) + \chi(3, 2, 1)] \\
& + 2(4 + \beta + 2\epsilon_{3a} - 2i\epsilon_{3b} + i\epsilon_{3c} - 2i\epsilon_{3d}) \chi(1, 3, 2) \\
& + 2(4 + \beta + 2\epsilon_{3a} + 2i\epsilon_{3b} - i\epsilon_{3c} + 2i\epsilon_{3d}) \chi(2, 1, 3) \\
& - 2(2 + 2i\epsilon_{3b} + i\epsilon_{3c})[\chi(1, 2, 4) + \chi(1, 4, 3)] \\
& - 2(2 - 2i\epsilon_{3b} - i\epsilon_{3c})[\chi(1, 3, 4) + \chi(2, 1, 4)] \\
& - 10[\chi(1, 2, 1, 2) + \chi(2, 1, 2, 1)] \\
& - (10 + i\epsilon_{3e} - i\epsilon_{3f})[\chi(2, 1, 2, 3) + \chi(2, 3, 2, 1)] \\
& - (10 - 8\zeta(3) - i\epsilon_{3e} + i\epsilon_{3f})[\chi(1, 2, 3, 2) + \chi(3, 2, 1, 2)] \\
& + \left(\frac{14}{3} + 8\zeta(3) + 2\epsilon_{3a} - 4i\epsilon_{3b} + i\epsilon_{3e}\right) [\chi(1, 3, 2, 3) + \chi(3, 1, 2, 1)] \\
& + \left(\frac{14}{3} - 4\zeta(3) + 2\epsilon_{3a} + 4i\epsilon_{3b} - i\epsilon_{3e}\right) [\chi(1, 2, 1, 3) + \chi(3, 2, 3, 1)] \\
& - 2(6 + \beta + 2\epsilon_{3a}) \chi(2, 1, 3, 2) \\
& + 2(9 + 2\epsilon_{3a})[\chi(1, 3, 2, 4) + \chi(2, 1, 4, 3)] \\
& - 2(4 + \epsilon_{3a} + i\epsilon_{3b})[\chi(1, 2, 4, 3) + \chi(1, 4, 3, 2)] \\
& - 2(4 + \epsilon_{3a} - i\epsilon_{3b})[\chi(2, 1, 3, 4) + \chi(3, 2, 1, 4)] \\
& - 10[\chi(1, 2, 3, 4) + \chi(4, 3, 2, 1)] ,
\end{aligned} \tag{2.29}$$

where we have also considered the possibility of more general similarity transformations as compared to  $\mathcal{N} = 4$  SYM. These are parameterized by two additional parameters  $\epsilon_{3e}$  and  $\epsilon_{3f}$  and they occur because the identities in (2.19) and (2.21) are no longer applicable. The details are worked out in appendix B. The dressing phase  $\beta$  and the coefficients  $\epsilon_{3a}, \dots, \epsilon_{3c}$  of similarity transformations in the scheme of  $\mathcal{N} = 1$  supergraphs are fixed by comparing the  $SU(2)$  subsector projection of the above result to the integrability-based expression. This is not the case for  $\epsilon_{3e}$  and  $\epsilon_{3f}$  that drop out in the projection and hence can be set to convenient values. With our choice we have made the rational numbers within the coefficients of  $\chi(1, 3, 2, 3) + \chi(3, 1, 2, 1)$  and  $\chi(1, 2, 1, 3) + \chi(3, 2, 3, 1)$  equal. This yields

$$\beta = 4\zeta(3) , \quad \epsilon_{3a} = -4 , \quad \epsilon_{3b} = -i\frac{4}{3} , \quad \epsilon_{3c} = i\frac{4}{3} , \quad \epsilon_{3e} = i\frac{4}{3} , \quad \epsilon_{3f} = i\frac{4}{3} , \tag{2.30}$$

Note that we have not fixed the coefficient  $\epsilon_{3d}$  in the scheme of Feynman diagrams in  $\mathcal{N} = 1$  superspace. The calculation would be tedious and unnecessary here, since the combination  $\chi(1, 3, 2) - \chi(1, 3, 2)$  of chiral functions that are conjugate to each other vanishes whenever applied to the states, and hence  $\epsilon_{3d}$  drops out.

### 3 Application to single-impurity states

We call chiral functions “connected” if adjacent entries in their arguments differ by  $\pm 1$ . These are the only chiral functions that do not vanish when acting on the one-impurity states in (1.6) and (1.9). For this reason the magnon dispersion relations can only depend on the connected chiral functions. Using the definition of a connected product of chiral functions in (A.8), we can reexpress the the first four orders of  $\mathcal{D}$  in (2.4) and (2.29) as

$$\begin{aligned}
\mathcal{D}_1 &= -2\chi(1) , \\
\mathcal{D}_2 &= -2[\chi(1)^2]_c , \\
\mathcal{D}_3 &= -4[\chi(1)^3]_c + 4i\epsilon_2[\chi(2, 1, 3) - \chi(1, 3, 2)] - 4\chi(1, 3) , \\
\mathcal{D}_4 &= -10[\chi(1)^4]_c \\
&\quad + 8\zeta(3)[-2\chi(1) + \chi(1, 2) + \chi(2, 1) \\
&\quad \quad - \chi(1, 2, 1) - \chi(2, 1, 2) + \chi(1, 2, 3, 2) + \chi(3, 2, 1, 2)] \\
&\quad - i(\epsilon_{3e} - \epsilon_{3f})[\chi(2, 1, 2, 3) + \chi(2, 3, 2, 1) - \chi(1, 2, 3, 2) - \chi(3, 2, 1, 2)] \\
&\quad + 8\left(\frac{23}{3} - \zeta(3)\right)\chi(1, 3) - 4\chi(1, 4) \\
&\quad + 2(4 + \beta + 2\epsilon_{3a} - 2i\epsilon_{3b} + i\epsilon_{3c} - 2i\epsilon_{3d})\chi(1, 3, 2) \\
&\quad + 2(4 + \beta + 2\epsilon_{3a} + 2i\epsilon_{3b} - i\epsilon_{3c} + 2i\epsilon_{3d})\chi(2, 1, 3) \\
&\quad - 2(2 + 2i\epsilon_{3b} + i\epsilon_{3c})[\chi(1, 2, 4) + \chi(1, 4, 3)] \\
&\quad - 2(2 - 2i\epsilon_{3b} - i\epsilon_{3c})[\chi(1, 3, 4) + \chi(2, 1, 4)] \\
&\quad + \left(\frac{14}{3} + 8\zeta(3) + 2\epsilon_{3a} - 4i\epsilon_{3b} + i\epsilon_{3e}\right)[\chi(1, 3, 2, 3) + \chi(3, 1, 2, 1)] \\
&\quad + \left(\frac{14}{3} - 4\zeta(3) + 2\epsilon_{3a} + 4i\epsilon_{3b} - i\epsilon_{3e}\right)[\chi(1, 2, 1, 3) + \chi(3, 2, 3, 1)] \\
&\quad - 2(6 + \beta + 2\epsilon_{3a})\chi(2, 1, 3, 2) \\
&\quad + 2(9 + 2\epsilon_{3a})[\chi(1, 3, 2, 4) + \chi(2, 1, 4, 3)] \\
&\quad - 2(4 + \epsilon_{3a} + i\epsilon_{3b})[\chi(1, 2, 4, 3) + \chi(1, 4, 3, 2)] \\
&\quad - 2(4 + \epsilon_{3a} - i\epsilon_{3b})[\chi(2, 1, 3, 4) + \chi(3, 2, 1, 4)] .
\end{aligned} \tag{3.1}$$

The terms with connected products appear naturally in the expansion of the  $\mathcal{N} = 4$  magnon dispersion relation [19, 24]. In particular, the first term in  $\mathcal{D}_4$  is consistent with a conjecture made in [19] that the rational connected parts of  $\mathcal{D}_m$  are given by

$$\frac{\Gamma(3/2)}{\Gamma(m+1)\Gamma(3/2-m)}(-4)^m[\chi(1)^m]_c . \tag{3.2}$$

The conjecture is nontrivial since it allows us to separate terms that would have been equivalent in the  $\beta$ -deformed or undeformed theory.

The next term in  $\mathcal{D}_4$  is a sum of connected chiral functions multiplied by  $8\zeta(3)$ . It vanishes when we project to the  $SU(2)$  subsector in  $\mathcal{N} = 4$  or the real  $\beta$ -deformed theory and hence is consistent with the dispersion relation (2.28) with  $h^2(g) = 4g^2$ . However, it contributes when two distinct types of scalar magnons approach each other, and hence is

associated with the scattering matrix for different magnons. In the complex  $\beta$ -deformed case it is also non-vanishing if a single magnon is within its interaction range and hence it starts to appear within the respective dispersion relation as a four-loop contribution to  $h^2(g)$  in (2.28).

We can now use (3.1) to determine the four-loop anomalous dimensions for particular operators.

### 3.1 Complex $\beta$ -deformation

The one-impurity operators in the complex  $\beta$ -deformations, where  $|q| \neq 1$  and  $h = 0$ , are eigenstates of  $\chi(1)$  with eigenvalue  $c(q, \bar{q})$ , where

$$c(q, \bar{q}) = \rho^2(1 - q)(1 - \bar{q}) = \frac{2(1 - q)(1 - \bar{q})}{(1 + q\bar{q})}, \quad (3.3)$$

and we have eliminated  $\rho$  by applying (2.6). Furthermore, the connected products of  $\chi(1)$  are products of  $c(q, \bar{q})$ . Using the relation in (A.4), we find

$$\gamma = [\sqrt{1 + 4g^2 c(q, \bar{q})} - 1] - 8\zeta(3)(1 - q\bar{q})^2 c^2(q, \bar{q}) + \mathcal{O}(g^{10}). \quad (3.4)$$

The part of  $\gamma$  inside the square bracket comes from the connected products in (3.1). The transcendental term comes from those connected chiral functions that are not included in the connected products. In the special case where  $\beta$  is real, the transcendental term drops out and we find  $c(q, \bar{q}) = 4\sin^2 \pi\beta$ . The remaining part of  $\gamma$  is the energy coming from (2.28) for one excitation with momentum shifted to  $p = 2\pi\beta$  by the twisted boundary conditions.

### 3.2 Cubic Leigh-Strassler deformation

In the cubic Leigh-Strassler deformation, where  $\rho \rightarrow 0$ ,  $\rho|h| = \sqrt{2}$ , the 3-string null operators are eigenstates of every chiral function. For  $\chi(1)$  the eigenvalue is  $-2M$ , where  $M$  is the number of pairs of adjacent fields with the same flavor. For all other chiral functions the eigenvalue is 0. Hence, the eigenvalue of the connected product  $[\chi(1)^m]_c$  is  $(-2)^m M$ . Inserting these expressions into (3.1), we find that the anomalous dimensions of these operators are  $\gamma_M = M\gamma_1$ , where

$$\gamma_1 = [\sqrt{1 + 8g^2} - 1] - 32\zeta(3)g^8 + \mathcal{O}(g^{10}). \quad (3.5)$$

Note that the four-loop result is consistent with a conjecture in [19] for the rational part of  $\gamma_1$ , where it was proposed that it would have the form in the square brackets to all orders in  $g$ , assuming that cancellations similar to (2.18) continue to hold<sup>1</sup>. However, there is also a transcendental contribution starting at four-loop order. A natural way to view this is that instead of  $g^2$ , the square root depends on a  $g$  dependent function  $\tilde{h}^2(g)$ , such that

$$\gamma_1 = \sqrt{1 + 8\tilde{h}^2(g)} - 1, \quad (3.6)$$

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<sup>1</sup>In [19] the cancellations were also shown to hold at five-loop order.

where

$$\tilde{h}^2(g) = g^2 - 8\zeta(3)g^8 + \dots \quad (3.7)$$

Without this correction the strong coupling behavior of  $\gamma_1$  would have been  $\gamma_1 \sim g$ , but with it  $\gamma_1$  likely increases with a smaller power of  $g$ .

## 4 Conclusions

The main result of this paper is the construction of the scalar part of the four-loop chiral dilatation operator (3.1). This construction is valid for any Leigh-Strassler deformation and it is complete for the closed scalar subsectors of these deformations. Completeness for an arbitrary deformation would require the contributions to  $\mathcal{D}$  involving the gauginos  $\psi_\alpha$ .

For the closed subsectors we have explicitly found the four-loop anomalous dimensions for the one impurity operators in the complex  $\beta$ -deformed theory and for *every* 3-string null operator in the cubic Leigh-Strassler theory. These anomalous dimensions have  $\zeta(3)$  terms that first appear at four loops<sup>2</sup>. These transcendental terms show up because they are present in the coefficients of connected chiral functions in  $\mathcal{D}_4$ .

From the perspective of  $\mathcal{N} = 4$  SYM or its integrable deformations, the  $\zeta(3)$  coefficients of the connected (or disconnected) chiral functions must trace back to the BES dressing phase [35]; the only place that transcendental terms appear in the all loop Bethe equations [28] is in the dressing phase and so it must be the source for these terms in the chiral dilatation operator. However, in the complex  $\beta$ -deformed theory, these same terms are associated with the dispersion relation, since they contribute to the anomalous dimensions for single impurity operators. This suggests that the dispersion relation for the complex  $\beta$ -deformed theory, or the anomalous dimensions of the 3-string null operators in the cubic deformation, are in principle derivable from the BES dressing phase. This assumes that the tuning mechanism as pictured in (2.16) and (2.18) continues to hold at higher loops, otherwise further corrections could spoil this relation.

For the immediate future, one can use the disentangling of the chiral functions to compute next to leading order wrapping effects, extending the analysis in [25, 26, 36–38]. It would also be interesting to find the complete four-loop chiral dilatation operator. For this it would be useful to find an extension of the chiral functions that could also include the gauginos.

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<sup>2</sup>The transcendental contributions do come one loop after those found in [32] for the  $\mathcal{N} = 2$  interpolating theory in [33, 34].

## A Relations between chiral functions

In the  $SU(2)$  subsector, the building block (2.6) of the chiral functions (2.5) reduces to

$$F_{ij}|_{SU(2)} = \rho^2(\mathbf{P} - \mathbb{1})_{ij} , \quad (\text{A.1})$$

where the deformed permutation and the identity on the r.h.s. are the ones given in (2.9) but projected into the subspace of two flavors by applying (2.10). Using the resulting expressions, one can check that the r.h.s. obeys the relation

$$(\mathbf{P} - \mathbb{1})_{nn+1}(\mathbf{P} - \mathbb{1})_{n+1n+2}(\mathbf{P} - \mathbb{1})_{nn+1} = q\bar{q}(\mathbf{P} - \mathbb{1})_{nn+1} . \quad (\text{A.2})$$

This relation has its origin in the fact that three adjacent elementary fields within the single trace of the composite operators of the  $SU(2)$  subsector carry maximally two different field flavors. The permutations then fulfill the relation

$$0 = \mathbb{1} - \mathbf{P}_{nn+1} - \mathbf{P}_{n+1n+2} + \mathbf{P}_{nn+1} \mathbf{P}_{nn+1} + \mathbf{P}_{n+1n+2} \mathbf{P}_{nn+1} - \mathbf{P}_{nn+1} \mathbf{P}_{n+1n+2} \mathbf{P}_{nn+1} , \quad (\text{A.3})$$

where the r.h.s. at vanishing  $\beta$ -deformation  $q = 1$  is nothing else than the total antisymmetrizer  $\epsilon_{i_n i_{n+1} i_{n+2}} \epsilon^{o_n o_{n+1} o_{n+2}}$  between three field flavors when expressed in terms of permutations. In the undeformed case, a respective relation was already worked out in [22]. When inserting (A.2) into the definition of the chiral functions (2.5), we find that in the  $SU(2)$  subsector they are reducible as

$$\chi(a_1, \dots, a_k, a, b, a, a_{k+4}, \dots, a_n)|_{SU(2)} = \rho^4 q\bar{q} \chi(a_1, \dots, a_k, a, a_{k+4}, \dots, a_n)|_{SU(2)} . \quad (\text{A.4})$$

For the general Leigh-Strassler deformation, it is very easy to check that the chiral functions fulfill

$$\begin{aligned} \chi(a_1, \dots, a_k, a, a, a_{k+3}, \dots, a_n) &= -\rho^2(1 + q\bar{q} + h\bar{h}) \chi(a_1, \dots, a_k, a, a_{k+3}, \dots, a_n) \\ &= -2 \chi(a_1, \dots, a_k, a, a_{k+3}, \dots, a_n) + \mathcal{O}(\kappa^6) , \end{aligned} \quad (\text{A.5})$$

where the second equality relies on (2.6). The previous relation between chiral functions is required in order to simplify the (noncommutative but associative) products of chiral functions. In order to define such products of chiral functions, we assume from now on that the length  $L$  is always sufficiently large, i.e.  $L \geq \kappa_a + \kappa_b - 1$ . Thereby,  $\kappa_a$ ,  $\kappa_b$  are the ranges, i.e. the numbers of legs involved in the chiral interactions, of the two chiral functions that are to be multiplied. In terms of the arguments of (2.5) the range is defined as

$$\kappa_a = \max_{a_1, \dots, a_n} - \min_{a_1, \dots, a_n} + 2 . \quad (\text{A.6})$$

We first introduce the commutator of two chiral functions that is given by

$$\begin{aligned} [\chi(a_1, \dots, a_n), \chi(b_1, \dots, b_p)] &= \sum_{\substack{\max_{b_1, \dots, b_p} - \min_{a_1, \dots, a_n} + 1 \\ b_1, \dots, b_p - \max_{a_1, \dots, a_n} - 1}} \chi(a_1 + k, \dots, a_n + k, b_1, \dots, b_p) \\ &\quad - \sum_{\substack{\max_{a_1, \dots, a_n} - \min_{b_1, \dots, b_p} + 1 \\ a_1, \dots, a_n - \max_{b_1, \dots, b_p} - 1}} \chi(b_1 + k, \dots, b_p + k, a_1, \dots, a_n) . \end{aligned} \quad (\text{A.7})$$



We call chiral functions “connected” if adjacent entries in their list of arguments never differ by more than  $\pm 1$ . For these chiral functions, we define the connected product as

$$[\chi(a_1 \dots, a_n) \chi(b_1, \dots, b_p)]_c = \sum_{k=b_1-a_n-1}^{b_1-a_n+1} \chi(a_1 + k, \dots, a_n + k, b_1, \dots, b_p) , \quad (\text{A.8})$$

such that the result is again a connected chiral function. It is understood that for multiple connected chiral functions appearing within  $[\dots]_c$ , the connected product is applied for each pairwise multiplication.

## B Similarity transformations

The representation of the dilatation operator is not unique, but it may be transformed by a change of the basis of operators that does not alter its eigenvalues. In this appendix, we work out such transformations. We include non-unitary cases that allow us to remove the anti-Hermitian contributions in the three-loop dilatation operator given in (2.4). The similarity transformations can be realized as

$$\mathcal{D}' = e^{-\chi} \mathcal{D} e^{\chi} = \mathcal{D} + \delta \mathcal{D} , \quad (\text{B.1})$$

where  $\chi$  is a linear combination of chiral functions. We demand that the transformations preserve the structural constraints coming from the underlying Feynman diagrams, i.e. at each loop order in the weak coupling expansion of  $\mathcal{D}$  the transformation must not generate contributions that involve chiral functions that can only appear at higher orders. This is guaranteed if the weak coupling expansion of  $\chi$  only contains those chiral functions that can be associated with Feynman diagrams at the considered order. In the  $SU(2)$  subsector of the  $\mathcal{N} = 4$  SYM theory the transformations are parameterized by one and four free parameters respectively in the three- and four-loop contribution to the dilatation operator. Here, we have to abandon the relation (2.19) and construct a more general transformation at four loops. The ansatz is given by

$$\begin{aligned} \chi = & g^2 i \delta_{11} \chi(1) + g^4 \left( i \delta_{21} \chi(1) + \frac{i}{2} \delta_{22} [\chi(1, 2) + \chi(2, 1)] \right) \\ & + g^6 \left( i \delta_{31} \chi(1) + \frac{i}{2} \delta_{32} [\chi(1, 2) + \chi(2, 1)] + i \delta_{33} \chi(1, 3) + \frac{i}{2} \delta_{34} [\chi(1, 2, 1) + \chi(2, 1, 2)] \right. \\ & + \frac{1}{2} (i \delta_{35} + \delta_{36}) \chi(1, 3, 2) + \frac{1}{2} (i \delta_{35} - \delta_{36}) \chi(2, 1, 3) \\ & \left. + \frac{i}{2} \delta_{37} [\chi(1, 2, 3) + \chi(3, 2, 1)] \right) , \end{aligned} \quad (\text{B.2})$$

where we have considered the fact that the chiral functions of the first two contributions in the last row are adjoint to each other, but the loop integrals of the respective diagrams are different.

Inserting this ansatz into (B.1) and expanding in powers of  $g$ , we respectively obtain

for the non-vanishing transformations at two and three loops

$$\begin{aligned}
\delta\mathcal{D}_3 &= -4i\epsilon_2[\chi(1, 3, 2) - \chi(2, 1, 3)] , \\
\delta\mathcal{D}_4 &= 2(2\epsilon_{3a} - 2i\epsilon_{3b} + i\epsilon_{3c} - 2i\epsilon_{3d})\chi(1, 3, 2) + 2(2\epsilon_{3a} + 2i\epsilon_{3b} - i\epsilon_{3c} + 2i\epsilon_{3d})\chi(2, 1, 3) \\
&\quad - 2i(2\epsilon_{3b} + \epsilon_{3c})[\chi(1, 2, 4) + \chi(1, 4, 3) - \chi(1, 3, 4) - \chi(2, 1, 4)] \\
&\quad - i(\epsilon_{3e} - \epsilon_{3f})[\chi(2, 1, 2, 3) + \chi(2, 3, 2, 1) - \chi(1, 2, 3, 2) - \chi(3, 2, 1, 2)] \\
&\quad + (2\epsilon_{3a} - 4i\epsilon_{3b} + i\epsilon_{3e})[\chi(1, 3, 2, 3) + \chi(3, 1, 2, 1)] \\
&\quad + (2\epsilon_{3a} + 4i\epsilon_{3b} - i\epsilon_{3e})[\chi(1, 2, 1, 3) + \chi(3, 2, 3, 1)] \\
&\quad - 4\epsilon_{3a}[\chi(2, 1, 3, 2) - \chi(1, 3, 2, 4) - \chi(2, 1, 4, 3)] \\
&\quad - 2(\epsilon_{3a} + i\epsilon_{3b})[\chi(1, 2, 4, 3) + \chi(1, 4, 3, 2)] \\
&\quad - 2(\epsilon_{3a} - i\epsilon_{3b})[\chi(2, 1, 3, 4) + \chi(3, 2, 1, 4)] , 
\end{aligned} \tag{B.3}$$

where the independent parameters read

$$\begin{aligned}
\epsilon_2 &= \frac{1}{2}(2\delta_{11} - \delta_{22}) , \\
\epsilon_{3a} &= -\frac{3}{2}\delta_{11}(2\delta_{11} - \delta_{22}) - \frac{1}{2}\delta_{36} , & \epsilon_{3b} &= 2\delta_{11} - \frac{1}{2}(\delta_{35} + \delta_{37}) , \\
\epsilon_{3c} &= -2\delta_{11} - \delta_{33} + \delta_{35} + \delta_{37} , & \epsilon_{3d} &= \delta_{21} + \delta_{22} - 12\delta_{11} - \frac{1}{2}\delta_{32} + \frac{3}{2}\delta_{35} + \delta_{37} , \\
\epsilon_{3e} &= \delta_{34} - 3\delta_{35} - \delta_{37} , & \epsilon_{3f} &= -3\delta_{35} . 
\end{aligned} \tag{B.4}$$

## C Integrals

All four-loop integrals  $I$  and their pole parts  $\mathcal{I}$  used in the text are given by

$$I_{4bb} = \text{Diagram 1} , \quad \mathcal{I}_{4bb} = \frac{1}{(4\pi)^8} \left( -\frac{1}{12\varepsilon^4} + \frac{1}{4\varepsilon^3} - \frac{1}{12\varepsilon^2} + \frac{1}{\varepsilon} \left( -\frac{5}{12} + \frac{1}{2}\zeta(3) \right) \right) ,$$

$$I_4 = \text{Diagram 2} , \quad \mathcal{I}_4 = \frac{1}{(4\pi)^8} \left( -\frac{1}{24\varepsilon^4} + \frac{1}{4\varepsilon^3} - \frac{19}{24\varepsilon^2} + \frac{5}{4\varepsilon} \right) ,$$

$$I_{4c} = \text{Diagram 3} , \quad \mathcal{I}_{4c} = \frac{1}{(4\pi)^8} \left( -\frac{1}{12\varepsilon^4} + \frac{5}{12\varepsilon^3} - \frac{13}{12\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{11}{12} - \frac{1}{2}\zeta(3) \right) \right) ,$$

$$I_{4w} = \text{Diagram 4} , \quad \mathcal{I}_{4w} = \frac{1}{(4\pi)^8} \left( -\frac{1}{24\varepsilon^4} + \frac{1}{4\varepsilon^3} - \frac{19}{24\varepsilon^2} + \frac{1}{\varepsilon} \left( \frac{5}{4} - \zeta(3) \right) \right) ,$$

$$I_{4\beta} = \text{Diagram 5} , \quad \mathcal{I}_{4\beta} = \frac{1}{(4\pi)^8} \left( -\frac{1}{12\varepsilon^4} + \frac{1}{3\varepsilon^3} - \frac{5}{12\varepsilon^2} - \frac{1}{\varepsilon} \left( \frac{1}{2} - \zeta(3) \right) \right) ,$$

$$I_{4bt} = \text{Diagram 6} , \quad \mathcal{I}_{4bt} = \frac{1}{(4\pi)^8} \left( -\frac{1}{2\varepsilon^2}\zeta(3) + \frac{1}{\varepsilon} \left( \frac{3}{2}\zeta(3) + \frac{\pi^4}{120} \right) \right) ,$$

$$I_{43t} = \text{Diagram 7} , \quad \mathcal{I}_{43t} = \frac{1}{(4\pi)^8} \left( -\frac{3}{2\varepsilon^2}\zeta(3) + \frac{1}{\varepsilon} \left( \frac{1}{2}\zeta(3) - \frac{\pi^4}{120} \right) \right) ,$$

$$I_{4t} = \text{Diagram 8} , \quad \mathcal{I}_{4t} = \frac{1}{(4\pi)^8} \frac{1}{\varepsilon} 5\zeta(5) ,$$

$$I''_{4t1} = \text{Diagram 9} , \quad \mathcal{I}''_{4t1} = \frac{1}{(4\pi)^8} \frac{1}{2\varepsilon} (\zeta(3) - 5\zeta(5)) ,$$

$$I''_{4t2} = \text{Diagram 10} = I''_{4t1} + \frac{1}{2}(I_4 - I_{4w} - I_{43t} + I_{4t}) ,$$

$$I_{4tr1} = \text{Diagram 11} \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta) ,$$

$$I_{4tr2} = \text{Diagram 12} \text{tr}(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta) .$$

(C.1)

We only need the sum of the last two integrals. This is much easier to work out than the individual integrals due to the symmetrization in pairs of indices that occurs after decomposing the numerator momentum with Lorentz index  $\beta$  in terms of the other two momenta at a vertex. One obtains the relation

$$I_{4\text{tr}1} + I_{4\text{tr}1} = 2I''_{4\text{t}2} - I_{4\text{b}t} + I_4 + I_{4\text{w}} + I_{4\beta} . \quad (\text{C.2})$$

Note also that  $\mathcal{I}_{4\text{w}}$  differs from  $\mathcal{I}_4$  only by an additional  $-\zeta(3)$  in the simple  $\frac{1}{\varepsilon}$  pole. Using then

$$2\mathcal{I}''_{4\text{t}1} + \mathcal{I}_{4\text{t}} = \frac{1}{(4\pi)^8} \frac{1}{\varepsilon} \zeta(3) , \quad (\text{C.3})$$

we can express  $\mathcal{I}_{4\text{w}}$  in terms of  $\mathcal{I}_4$  as

$$\mathcal{I}_{4\text{w}} = \mathcal{I}_4 - 2\mathcal{I}''_{4\text{t}1} - \mathcal{I}_{4\text{t}} . \quad (\text{C.4})$$

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